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Prove that for any positive real numbers a, b, c

$$(\sqrt{3}-1)\sqrt{ab+bc+ca}+3\sqrt{\frac{abc}{a+b+c}} \le a+b+c.$$

Solution by Arkady Alt, San Jose, California, USA.

Assume $a+b+c=1$ (due homogeneity of the inequality) and denote $p := ab+bc+ca, q := abc.$

Then $p=(ab+bc+ca)(a+b+c) \ge 9abc = 9q,\ p = ab+bc+ca \le \frac{(a+b+c)^2}{3} = \frac{1}{3}$ and original inequality becomes $(\sqrt{3}-1)\sqrt{p}+3\sqrt{q} \le 1 \iff 1-(\sqrt{3}-1)\sqrt{p}-3\sqrt{q} \ge 0.$

We have $1-(\sqrt{3}-1)\sqrt{p}-3\sqrt{q} \ge 1-(\sqrt{3}-1)\sqrt{p}-3\sqrt{\frac{p}{9}} = 1-(\sqrt{3}-1)\sqrt{p}-\sqrt{p} =$

$1-\sqrt{3p} = \frac{1-3p}{1+\sqrt{3p}} \ge 0.$